J Glob Optim (2008) 40:489–494
DOI 10.1007/s10898-007-9250-1

Localization of minimax points

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Received: 22 September 2007 / Accepted: 28 September 2007 / Published online: 20 November 2007

Abstract In their proof of Gilbert–Pollak conjecture on Steiner ratio, Du and Hwang (Proceedings 31th FOCS, pp. 76–85 (1990); Algorithmica 7:121–135, 1992) used a result about localization of the minimum points of functions of the type $\max_{y\in Y} f(\cdot, y)$. In this paper, we present a generalization of such a localization in terms of generalized vertices, when we minimize over a compact polyhedron, and Y is a compact set. This is also a strengthening of a result of Du and Pardalos (J. Global Optim. 5:127–129, 1994). We give also a random version of our generalization.

Keywords Minimax · Gilbert–Pollak conjecture · Steiner ratio · g-Vertex · Steiner tree · Spanning tree

1 Introduction

We consider a classical minimax problem

$$\min_{x\in X}\max_{y\in Y} f(x, y), \tag{1}$$

where X is an appropriate set and Y is a compact set. We are interested in determination of a subset $B \subset X$ (usually finite) such that

$$\min_{x\in B}\max_{y\in Y} f(x, y) = \min_{x\in X}\max_{y\in Y} f(x, y). \tag{2}$$

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This problem, for specific f (defined by interpolation problems), goes back to Chebyshev, who contributed by it to the foundations of best approximation theory.

For more general f, appropriate determination of B in (1) is given by Du and Hwang [1,5, pp. 339–367], as follows:

Theorem 1 (Du–Hwang) *Suppose that f_i, $i = 1, \ldots, m$, are continuous and concave functions on the compact polyhedron*

$$X = \{x \in \mathbb{R}^n : x^T a_j \leq b_j, j = 1, \ldots, k\}.$$

Then the minimum value of the function

$$g(x) = \max_{i=1,\ldots,m} f_i(x)$$

over X is achieved at a g-vertex, i.e., at a point x^* with the property that there is no $y \in X$ such that at least one of the sets $M(x^*)$ and $J(x^*)$ is a proper subset of $M(y)$ and $J(y)$ respectively, where

$$J(x) := \left\{ j \in \{1, \ldots, k\} : x^T a_j = b_j \right\},$$

and

$$M(x) := \{i \in \{1, \ldots, m\} : g(x) = f_i(x)\}.$$

With the above theorem, Du and Hwang [1,2] proved the Gilbert–Pollak conjecture (1966) that the *Steiner ratio* in the Euclidean plane is $\sqrt{3}/2$ (see also [2,5]). Let $G(V, E)$ be a network with a set of edges E and a set of vertices V. Let M be a subset of V. Let us recall some important notions.

- *Steiner minimum tree (SMT)* on the finite point set M—the shortest network (tree) interconnecting the points in this set. It may contain vertices not in M.
- Minimum spanning tree (MST) on M—a shortest spanning tree with vertex set M.
- Gilbert–Pollak conjecture: $\inf_{M \subset \mathbb{R}^2} \frac{L_s(M)}{L_m(M)} = \sqrt{3}/2$,

  where $L_s(M)$ and $L_m(M)$ are the lengths of $SMT(M)$ and $MST(M)$ respectively.

An interior point x of X is a g-vertex iff $M(x)$ is maximal. In general, for any g-vertex, there exists an extreme subset Y of X such that $M(x)$ is maximal over Y. A point x of X is called a *critical point*, if there exists an extreme set Y such that $M(x)$ is maximal over Y. Thus, every g-vertex is a critical point. However, the inverse is not true.

Du and Pardalos [4] generalized the above theorem for the case when the index set is infinite—a compact set, as follows.

Theorem 2 (Du, Pardalos) *Let f: $X \times I \to \mathbb{R}$ be a continuous function, where X is a compact polyhedron in $\mathbb{R}^m$ and I is a compact set in $\mathbb{R}^n$. Let $g(x) = \max_{y \in Y} f(x, y)$. If $f(x, y)$ is a concave function with respect to x, then the minimum value of g over X is achieved at some critical point.*

In this paper we strengthen the results of Du–Pardalos and Du–Hwang, showing that the minimum point in (1) is achieved at a generalized vertex (not only at a critical point), when Y is compact, and X is a compact polyhedron. We give a random version of this strengthening.

2 Compact polyhedrons

Here we give necessary and sufficient conditions for compactness of a polyhedron

$$P = \{x \in \mathbb{R}^n : x^T a_j \leq b_j,\ j = 1, \dots, k\}.$$

Compact polyhedrons are used in the next theorems, so such a characterization is useful.

Proposition 3 *The polyhedron P is compact if and only if*

$$0 \quad \textit{belongs to the interior of} \quad co\{a_j\}_{j=1}^k. \tag{3}$$

Proof Note first that (3) implies that $k \geq n + 1$ (otherwise $co\{a_j\}_{j=1}^k$ will have no interior points) and that the span of $\{a_j\}_{j=1}^k$ is all $\mathbb{R}^n$. Without loss of generality, we may assume that $b_j \geq 0$ (replacing P with $P - p_0$ and b_j with $b_j - p_0^T a_j$ if necessary, where p_0 is any element of P).

(a) Assume that (3) is satisfied and P is unbounded. Then there is a sequence $x_i \in P$ such that $\|x_i\| \to \infty$. By compactness of the unit sphere, there is a subsequence $\{x_{i_r}\} \subset \{x_i\}$ such that $\frac{x_{i_r}}{\|x_{i_r}\|}$ converges to some x_0 with $\|x_0\| = 1$. Since P is closed, $x_0 \in P$ and

$$x_0^T a_j \leq 0, \quad j = 1, \dots, k. \tag{4}$$

By (3) and Carathéodory's theorem there are positive α_r, $r = 1, \dots, n+1$, and indices $j_1, \dots, j_{n+1}$ such that the span of $\{a_{j_r}\}_{r=1}^{n+1}$ is $\mathbb{R}^n$ and

$$\sum_{r=1}^{n+1} \alpha_r a_{j_r} = 0. \tag{5}$$

There is an index r_0 such that $x_0^T a_{j_r} < 0$; otherwise x_0 should be orthogonal to all a_{j_r}, $r = 1, \dots, n+1$, which is a contradiction with the fact that the span of $\{a_{j_r}\}_{r=1}^{n+1}$ is all $\mathbb{R}^n$.

Multiplying (4) with α_{j_r}, $r = 1, \dots, n+1$, and summing, we obtain a contradiction.

(b) Assume that P is compact and (3) is not satisfied. By separation theorem, we can separate 0 and $co\{a_j\}_{j=1}^k$, so there is a $0 \neq y \in \mathbb{R}^n$ such that

$$y^T a_j \leq 0 \leq b_j, \quad j = 1, \dots, k.$$

So, the ray $\{\beta y : \beta > 0\}$ will belong to P, a contradiction with the compactness of P.

□

3 Localization of minimax points

Define

$$J(x) := \left\{ j \in \{1, \dots, k\} : x^T a_j = b_j \right\}$$

and

$$M(x) := \{y \in Y : g(x) = f(x, y)\}.$$

The point $\hat{x}$ will be called a *generalized vertex* (in short *g-vertex*) if there is no $z \in X$ such that the set $J(\hat{x}) \cup M(\hat{x})$ is a proper subset of $J(z) \cup M(z)$.

Theorem 4 (General localization theorem) *Suppose that Y is a compact topological space, $X \subset \mathbb{R}^n$ is a compact polyhedron, $f: X \times Y \to \mathbb{R}$ is a continuous function and $f(\cdot, y)$ is concave for every $y \in Y$. Define $g(x) = \max_{y \in Y} f(x, y)$. Then the minimum of g over X is attained at some generalized vertex $\hat{x}$.*

Proof Let x^* be a minimum point for g. We will prove, applying the Zorn lemma, that there exists a g-vertex $\hat{x}$ satisfying $M(x^*) \subset M(\hat{x})$ and $J(x^*) \subset J(\hat{x})$. Let us consider the partial ordering $\preceq$ on the set

$$S = \{x \in X : M(x^*) \subseteq M(x), J(x^*) \subseteq J(x)\}$$

defined by

$$x \preceq y \Leftrightarrow (M(x) \cup J(x)) \subseteq (M(y) \cup J(y)).$$

Let $\{x_i\}_{i=1}^{\infty}$ be a linearly ordered subset of $(S, \preceq)$, i.e.,

$$M(x^*) \subseteq M(x_1) \subseteq M(x_2) \subseteq \cdots \subseteq M(x_n) \ldots$$
$$J(x^*) \subseteq J(x_1) \subseteq J(x_2) \subseteq \cdots \subseteq J(x_n) \ldots$$

Since X is compact, there exists a subsequence $\{x_{i_k}\}$ converging to some x'. Since Y is compact, the function $x \mapsto \max f(x, Y)$ is continuous. So, if $y \in M(x_n)$ for some n, then $y \in M(x_{k_i})$ for sufficiently large i, therefore

$$\max f(x', Y) = \lim_{k \to \infty} \max f(x_{i_k}, Y) = \lim f(x_{k_i}, y) = f(x', y),$$

which means $y \in M(x')$ or $M(x_n) \subseteq M(x')$ for every n. Similarly we can prove that $J(x_n) \subseteq J(x')$ for every n. Thus, x' is an upper bound for $\{x_n\}_{n=1}^{\infty}$ with respect to $\preceq$. By the Zorn lemma, $(S, \preceq)$ has a maximal element $\hat{x}$, i.e., $\hat{x}$ is a g-vertex.

In particular, $M(x^*) \subseteq M(\hat{x})$ and $J(x^*) \subseteq J(\hat{x})$. The latter inclusion implies that $\hat{x}^T a_j = b_j, \forall j \in J(x^*)$. There is $\lambda_0 > 0$ such that

$$x(\lambda) := x^* + \lambda(x^* - \hat{x}) \in P \text{ for every } \lambda \in (0, \lambda_0).$$

We will prove that $\hat{x}$ is a minimum point of g. Assume the contrary. Consider the cases:

Case 1 There exists $\lambda \in (0, \lambda_0)$ such that $M(x(\lambda)) \cap M(x^*) \neq \emptyset$.

Take $y \in M(x(\lambda)) \cap M(x^*)$. Then

$$x^* = \frac{\lambda}{1+\lambda}\hat{x} + \frac{1}{1+\lambda}x(\lambda).$$

By concavity of $f(., y)$ we have

$$f(x^*, y) \geq \frac{\lambda}{1+\lambda} f(\hat{x}, y) + \frac{1}{1+\lambda} f(x(\lambda), y) > f(x^*, y),$$

a contradiction.

Case 2 $M(x(\lambda)) \cap M(x^*) = \emptyset$ for every $\lambda \in (0, \lambda_0)$.

Let $y(\lambda) \in M(x(\lambda))$. By compactness, there exists a sequence $\{\lambda_k\}$ converging to zero such that $y(\lambda_k)$ converges to some $y^* \in Y$. Since $g(x(\lambda_k)) = f(x(\lambda_k), y(\lambda))$, by continuity we obtain $g(x^*) = f(x^*, y^*)$, so $y^* \in M(x^*)$. Since $M(x^*) \subseteq M(\hat{x})$, it follows that $f(\hat{x}, y^*) = g(\hat{x}) > g(x^*)$. Therefore, for sufficiently large k,

$$f(\hat{x}, y(\lambda_k)) > g(x^*).$$

Since $y(\lambda_k) \in M(x(\lambda_k))$ and $y(\lambda_k) \notin M(x^*)$, we have

$$f(x(\lambda_k), y(\lambda_k)) = g(x(\lambda_k)) \geq g(x^*),$$

and

$$f(x^*, y(\lambda_k)) < g(x^*).$$

Thus

$$f(x^*, y(\lambda_k)) < \min\{f(x(\lambda_k), y(\lambda_k)), f(\hat{x}, y(\lambda_k))\},$$

which is a contradiction with the concavity of $f(x^*, .)$. □

Remark The above theorem generalizes a theorem of Du and Pardalos [4]. Namely, they prove that the minimum of g is attained at a critical point $\hat{x}$ characterized by the following:

(*) There exists an extreme subset Z of X such that $\hat{x} \in Z$ and the set $M(\hat{x}) = \{y \in Y : g(\hat{x}) = f(\hat{x}, y)\}$ is maximal over Z.

It is easy to see that every g-vertex is a critical point, but the inverse is not true (see for instance, [5], Remark 2).

4 Random g-vertices

In this section (Ω, Σ) is a measure space, i.e., Ω is a set and Σ is a σ algebra on Ω. A (multi)function $F : \Omega \to \mathbb{R}^n$ is measurable iff $F^{-1}(B)$ is measurable for each closed subset B of $\mathbb{R}^n$ (i.e., $F^{-1}(B) := \{\omega \in \Omega : F(\omega) \cap B \neq \emptyset\} \in \Sigma$). For a comprehensive description of measurable relations, see for instance [6].

Theorem 5 (Random g-vertices) *Suppose that X in Theorem* 1 *is a compact random polyhedron, i.e., it depends on ω and is given by*

$$X(\omega) = \{x \in E : x^T a_j(\omega) \leq b_j(\omega) \quad j = 1, \ldots, k\},$$

where $a_j : \Omega \to \mathbb{R}^n$ and $b_j : \Omega \to \mathbb{R}$ are measurable functions. Then there exists a measurable function $\omega \mapsto \hat{x}(\omega)$ with $\hat{x}(\omega)$ being a generalized vertex of $X(\omega)$ for every $\omega \in \Omega$.

Proof First we prove that there is a measurable minimizer $x^*(\omega)$ of the minimization problem

$$\text{minimize} \quad \max f(\omega, x, Y) \tag{6}$$
$$\text{subject to} \quad x \in X(\omega) \tag{7}$$

The proof of this fact is routine and is based on known measurability theorems, contained, for instance, in [6].

Next, following the proof of Theorem 4, we prove that there is a g-vertex $\hat{x}(\omega)$, which is a measurable function on ω.

Let us consider the partial ordering $\preceq$ on the set of all measurable selections of the multifunction $X(\omega)$ i.e.,

$$S = \{x : \Omega \to X(\omega) : M(x^*(\omega)) \subseteq M(x(\omega)), J(x^*(\omega)) \subseteq J(x(\omega)), \quad \forall \omega \in \Omega\}$$

defined by

$$x \preceq y \Leftrightarrow (M(x(\omega)) \cup J(x(\omega))) \subseteq (M(y(\omega)) \cup J(y(\omega))) \quad \forall \omega \in \Omega.$$

Let $\{x_i\}_{i=1}^{\infty}$ be a linearly ordered subset of $(S, \preceq)$, i.e.,

$$M(x^*(\omega)) \subseteq M(x_1(\omega)) \subseteq M(x_2(\omega)) \subseteq \cdots \subseteq M(x_n(\omega)) \ldots \quad \forall \omega \in \Omega,$$

$$J(x^*(\omega)) \subseteq J(x_1(\omega)) \subseteq J(x_2(\omega)) \subseteq \cdots \subseteq J(x_n(\omega)) \ldots \quad \forall \omega \in \Omega.$$

Consider the set

$$X_n(\omega) = \left\{ x \in X(\omega) : \left(x + \frac{1}{n} B \right) \cap \{x_k(\omega)\}_{k=n}^{\infty} \neq \emptyset \right\}.$$

Since

$$X_n(\omega) = \bigcup_{k=n}^{\infty} \left(x_k(\omega) + \frac{1}{n} B \right),$$

we have

$$X(\omega) \setminus X_n(\omega) = \bigcap_{k=n}^{\infty} \left(X(\omega) \Big\backslash \left(x_k(\omega) + \frac{1}{n} B \right) \right). \tag{8}$$

Thus, by Theorem 4.1 of [6], the multifunction $\omega \mapsto X(\omega) \backslash X_n(\omega)$ is measurable, and by Theorem 4.5 of [6], the function $\omega \mapsto X_n(\omega)$ is measurable. Putting $C(\omega) = \bigcap_{n=1}^{\infty} \overline{X_n(\omega)}$ and applying again Theorem 4.5 of [6], we obtain that the multifunction $\omega \mapsto C(\omega)$ is measurable. Now by the Kuratowski, Ryll–Nardzewski selection theorem (see [6], Theorem 5.1), there is a measurable selection $x'(\omega) \in C(\omega), \forall \omega \in \Omega$. Since $C(\omega)$ is the set of all cluster points of the sequence $\{x_n(\omega)\}_{n=1}^{\infty}$, the point $x'(\omega)$ is a cluster point too, depending in a measurable way on ω. As in the proof of Theorem 4, we obtain that $x'(\omega)$ is an upper bound for $\{x_n(\omega)\}_{n=1}^{\infty}$ with respect to $\preceq$. By the Zorn lemma, $(S, \preceq)$ has a maximal element $\hat{x}(\omega)$ (measurable on ω), i.e., $\hat{x}(\omega)$ is a measurable g-vertex.

Further we prove exactly as in Theorem 4 that $\hat{x}(\omega)$ is a minimum point of the function $\max f(\omega, \cdot, Y)$ over $X(\omega)$, which completes the proof of the theorem. □

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